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Maximum Theoretical Angular Accuracy of Planar and Linear Arrays of Sensors

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A study has been made of the maximum theoretical accuracy in the angular location of a radiating object that is achievable by using a planar or linear array. The elements are assumed to have identical radiation patterns and the complex voltages observed at their ports are assumed to be subject to phase measurement errors, having normal probability density. An optimum scheme for the statistical extraction of the parameters defining the direction is

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established. It consists of combining the observed phases linearly with weights depending upon the element locations. It is shown that the presence of thermal noise, for sufficiently high signal to noise ratio, does not change the structure of the estimator. Comparison with conventional multiple interferometric techniques indicates the superiority of the proposed scheme. Finally, a limited numerical study on a small linear array vertically located on a reflecting terrain is performed. Although in such a situation the scheme proposed is not the theoretical optimum, it leads to errors that, for most directions of the target, are smaller than those found for the same array when using conventional multiple interferometer techniques.

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1. INTRODUCTION

The objectives of this paper are:

To establish an upper bound on the theoretical accuracy in the angular location of a radiating object obtainable with an array of identical sensors subject to instrumental errors, in the absence of angular interference;

The determination of an optimum processing scheme, which, under certain assumptions, meets the bound;

A limited numerical investigation of a particular array in the reasonably realistic situation for which terrain specular reflection is present.

The question of the errors in determining the angular coordinates of an object located in the far zone of a planar or linear array of antennas has been studied, among other applications, for radar antennas[1-3] and for multiple interferometers for landing systems.[4] In the first area of applications mentioned, the array

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1. Rondinelli, L.A. (1959) Effects of random errors on the performance of antenna arrays of many elements, 1959 IRE Nat. Conv. Record, pt 1, pp 174-189.

2. Allen, J.L. (1963) The Theory of Array Antennas, MIT Lincoln Laboratory Tech Report No. 323, pp 69-70.

3. Carver, Keith R., Cooper, W.K., and Stutzman, W.L. (1973) Beam pointing errors of planar phased arrays, IEEE Transaction on Antennas and Propagation, AP-21, No. 2:199-202.

4. Danville, A.R. and Moore, S.R. (1975) Design method for interferometer arrays, Proceedings of the IEE (British), 122, No. 6:605-608.

element outputs are linearly combined in the feed structure in order to provide a small set of "observable" voltages (the outputs of the sum and difference ports of a monopulse antenna system) from which the unknown target angular coordinates are to be extracted. In the second area of application the angular position is obtained by averaging phase measurements pertinent to interferometric baselines of different lengths.

The two techniques are, of course, totally different and are applied also in totally different circumstances. The first technique uses a "conventional" and hence linear antenna. The second instead embodies processing schemes which are highly nonlinear. Also in most cases the second technique is applied to locate only cooperative targets, (as in microwave landing systems) having a transponder on board. Thus the density of the received power incident on the array can be orders of magnitude greater than for the radar case, thereby making considerations of antenna gain totally different in the two cases. In this work the question of the angular accuracy of a system of sensors having equal radiation patterns is reexamined from a different and more general viewpoint. Rather than analysing a particular processing scheme assumed a priori, an "unstructured" approach is adopted based on statistical inference considerations.[5] An estimation criterion rather than a processing scheme is established a priori for the extraction of the angular parameters from the phases of the complex voltages observed at the element ports (the amplitudes carrying no angular information). The processing scheme is instead obtained from the set of equations that are the analytical expression of the criterion. The maximum likelihood criterion, with its well-known desirable statistical properties, is the one chosen.[6] If the probability densities of the phase errors of the element voltages are normal, the estimator, that is, the processing scheme of the observables, consists of linearly combining the phases of the element outputs with weights depending upon the element coordinates, referred to the geometrical "center of mass" of the array (Section 3). For normal density of phase errors the estimator is "optimal", in the sense of having the smallest possible rms estimation errors among all possible estimators that are "unbiased", that is, without systematic errors. It is, in fact, shown that under the hypothesis of gaussian probability densities for uncorrelated errors in the measurements of element phases, the rms estimation errors meet a Cramer-Rao bound derived in Appendix C. Thus the proposed algorithm provides the maximum possible angular accuracy (for unbiased estimation), which cannot be improved no matter how sophisticated the design of the processing scheme.

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5. Van Trees, H.L. (1968) Detection Estimation and Linear Modulation Theory, John Wiley and Sons, Chap. 1.

6. Van Trees, H.L. (1968) op. cit., Chap. 2.

In this paper planar and linear arrays, with elements in general non-equally spaced, are considered. As a numerical application, the performance of a linear array, having the elements spaced in geometric progression (with the minimum distance between elements small enough to eliminate ambiguity in the phase readings) with a total length of 24 wavelengths, is discussed. The accuracy found is substantially greater than that obtained for the same geometry and phase errors by using multiple interferometer techniques (Section 4). For the same array vertically located on a reflecting terrain, the degradation of performance due to multipath is also examined (Section 5). For this situation the estimation procedure proposed is, of course, not the theoretical optimum. In fact, not only phases but also relative amplitudes of element output voltages should be used because they also carry angular information. Also in this case the phase "errors" due to multipath are highly correlated and difficult to characterize statistically. Nevertheless, for the particular case considered, the performance of the scheme discussed here is substantially better than that obtainable through a conventional multiple interferometer technique. This leads to the conjecture, to be verified through analysis and/or simulation, that the scheme is in general superior to the multiple interferometer technique also in the realistic case of presence of reflection.

Finally, it is shown that the inclusion in the model of thermal noise (assumed absent when obtaining the above results) does not affect the structure of the estimator (but, of course, affects the accuracy) (Section 6).

2. STATEMENT OF THE PROBLEM AND NOTATION

The discussion is conducted in terms of planar arrays. The results for the linear simpler case are then obtained from those pertinent to the planar case through appropriate simplifications. Our main objective is to assess the effects of instrumentation errors, that is the errors affecting the voltages at the element ports. Thus (at least in the first part of this study) thermal noise, introduced in the voltage measurements, that is noise at the element receivers, will be considered negligible with respect to the incoming signal. The elements of the array, or rather their radiation patterns, are assumed identical and identically oriented. Since (at least for the time being) we restrict our considerations to the angular location of a single target in the absence of angular interferences, no angular information is contained in the amplitudes of the element output voltages. Thus only the set of the phases of the element voltages constitutes the observables to be used for the statistical extraction of the angular coordinates.

Let x_i, y_i be the coordinates of the ith sensor in the x, y plane of the array, the total number of elements being N. It proves convenient to introduce the coordinates of the geometrical "Center of Mass" of the antenna elements:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \, . \qquad (1)$$

$$\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i \, . \qquad (2)$$

The following geometrical quantities are also introduced for formal simplification of our results:

$$M_x = \sum_{i=1}^{N} (x_i - \bar{x})^2 \, , \qquad (3)$$

$$M_y = \sum_{i=1}^{N} (y_i - \bar{y})^2 \, , \qquad (4)$$

$$M_{xy} = \sum_{i=1}^{N} \left[(x_i - \bar{x})(y_i - \bar{y}) \right] \, . \qquad (5)$$

Let α, β be the unknown cosines of the direction of incidence with respect to the x, y axis. If $k = 2\pi/\lambda$ is the free-space propagation constant, the propagation vector of the incident wave is

$$\underline{k} = k\left(\alpha \hat{x} + \beta \hat{y} + \sqrt{1 - \alpha^2 - \beta^2}\, \hat{z}\right),$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors in the directions of the rectangular axes.

The phase ψ_i observed at the output of the ith element is expressed at the sum of three terms:

$$\psi_i = k(\alpha x_i + \beta y_i) + \mu + \delta_i \, . \qquad (6)$$

The first term depends upon the element position and contains the unknown parameters. The second term μ is a reference phase, interpreted as the phase that one would measure at the output of an element assumed located at the origin of the coordinate system in the absence of measurement errors. The quantities δ_i denote the instrumental errors in the phase measurement. They may depend upon mechanical tolerances of the elements, quantization in the phase measuring device, and so forth, and are constant in time. The important point is that the δ_i's are unknown, except for certain statistical properties, and can be treated as random variables.

We will make several reasonable assumptions about the δ_i's. The probability density is the same for all elements:

$$p_i(\delta_i) = p(\delta_i) ,$$

that is, independent of i. The δ_i's have zero mean:

$$E[\delta_i] = \int p(\delta_i)\, d\delta_i = 0 , \tag{7}$$

where E[ ] represents the statistical average operator. Also errors in different elements are statistically independent, that is, with self-explanatory notation:

$$p_t(\delta_1, \delta_2 \ldots \delta_N) = p(\delta_1)\, p(\delta_2) \ldots p(\delta_N) , \tag{8}$$

so that:

$$E[\delta_i\, \delta_k] = \sigma_\delta^{\,2}\, \delta_{ik} . \tag{9}$$

where σ_δ is the rms value of the phase error and δ_{ik} is Kronecker's delta, equal to 1 for $i = k$, equal to zero for $i \neq k$. Because of (6), Eq. (8) can be written as a conditional probability density for the set of phases $\{\psi_i\}$ for given values of the parameters α, β, μ:

$$p_t(\psi_1, \psi_2 \ldots \psi_N \mid \alpha, \beta, \mu) = \prod_{i=1}^{N} p\left[\psi_i - \mu - k(\alpha\, x_i + \beta\, y_i)\right] . \tag{10}$$

If the function $p(\cdot)$ in Eq. (8) is known, Eq. (10) summarizes all the statistical information available on α, β, given the set of observables $\{\psi_i\}$. In the particular case of normally distributed phase errors,

$$p(\delta_i) = \frac{1}{\sqrt{2\pi}\,\sigma_\delta} \exp\left(-\frac{\delta_i^{\,2}}{2\sigma_\delta^{\,2}}\right) . \tag{11}$$

We want to establish a statistically satisfactory estimation algorithm for the extraction from the set of observables ψ_i of the unknown parameters α, β.

3. MAXIMUM LIKELIHOOD METHOD OF PHASE FRONT SLOPE EVALUATION

3.1 Theory

Under the idealized assumption of absence of directional or diffused interference the phase front of the wave incident on the array is planar. Since all angular information is contained in the slope of the phase front, an intuitively appealing procedure for the extraction of α, β, consists of measuring the phases of the element voltages then combining them linearly to extract the components of the phase slope. If the weights used in the linear combination are properly chosen, this procedure implements the Maximum Likelihood Estimation (MLE) criterion under the assumption (11) of gaussian probability density for phase errors. Furthermore, under the same assumption, it yields an unbiased estimator having the minimum possible rms errors.

We recall that the MLE consists in assuming that, for a given set of observables, the most likely values of the unknown parameters are those for which the probability of errors is maximum. Hence, following the standard procedure, we take the derivatives of the logarithm of the conditional probability densities (10), and equate them to zero.[6] The estimates of α, β, μ, are those values $\hat{\alpha}$, $\hat{\beta}$, $\hat{\mu}$ for which the equation

$$\left.\frac{\partial \log p_t(\psi_1, \psi_2, \ldots \psi_N \mid \alpha, \beta, \mu)}{\partial \alpha}\right|_{\alpha=\hat{\alpha},\ \beta=\hat{\beta},\ \mu=\hat{\mu}} = 0 \tag{12}$$

holds, together with the two parallel equations, obtained by replacing $\partial\alpha$ in (12) with $\partial\beta$ and $\partial\mu$, respectively. Assume normal densities for the δ_i's. Thus from (10) and (12) we obtain:

$$\sum_{i=1}^{N} (\psi_i - k\hat{\alpha}x_i - k\hat{\beta}\, y_i - \hat{\mu})\, x_i = 0\ , \tag{13}$$

$$\sum_{i=1}^{N} (\psi_i - k\hat{\alpha}x_i - k\hat{\beta}\, y_i - \hat{\mu})\, y_i = 0\ , \tag{14}$$

$$\sum_{i=1}^{N} (\psi_i - k\hat{\alpha}x_i - k\hat{\beta}\, y_i - \hat{\mu}) = 0\ . \tag{15}$$

By eliminating in (13-15) the estimate $\hat{\mu}$ of the parameter μ (of no interest) the explicit forms of the estimators $\hat{\alpha}$ and $\hat{\beta}$ are obtained (Appendix A):

$$\hat{\alpha} = \frac{1}{k} \frac{\sum_{i=1}^{N} \left[\psi_i \quad (x_i - \bar{x})\, M_y - (y_i - \bar{y})\, M_{xy}\right]}{M_x M_y - M_{xy}^2} \quad , \tag{16}$$

$$\hat{\beta} = \frac{1}{k} \frac{\sum_{i=1}^{N} \left[\psi_i\, (y_i - \bar{y})\, M_x - (x_i - \bar{x})\, M_{xy}\right]}{M_x M_y - M_{xy}^2} \quad . \tag{17}$$

In Appendix B it is shown that

$$E[\hat{\alpha}] = \alpha, \qquad E[\hat{\beta}] = \beta, \tag{18}$$

that is, the estimates (16-17) are unbiased. In Appendix B the quadratic errors of estimations are shown to be:

$$\sigma_\alpha{}^2 = \frac{\sigma_\delta{}^2}{k^2} \frac{M_y}{M_x M_y - M_{xy}^2}, \tag{19}$$

$$\sigma_\beta{}^2 = \frac{\sigma_\delta{}^2}{k^2} \frac{M_x}{M_x M_y - M_{xy}^2}. \tag{20}$$

In Appendix C it is shown that under the hypothesis of normal densities for the phase measurement errors the rms estimation errors (19-20) are the minimum possible for unbiased estimators. This means that the algorithm (16-17) is optimal for yielding the maximum possible angular accuracy in the absence of systematic errors.

The cross correlation of the estimators $\hat{\alpha}$ and $\hat{\beta}$ is (Appendix B):

$$\sigma_{\alpha\beta} = \frac{\sigma_\delta{}^2}{k^2} \frac{M_{xy}}{M_x M_y - M_{xy}^2}. \tag{21}$$

From (19-21) the joint probability density for $\hat{\alpha}$ and $\hat{\beta}$ is easily established:

$$p(\hat{\alpha}, \hat{\beta}) = \frac{k}{2\pi\, \sigma_\delta{}^2} \sqrt{M_x M_y - M_{xy}^2}\, \exp \left\{ - \frac{k^2}{2\sigma_\delta{}^2} \left[M_x(\alpha - \hat{\alpha})^2 + M_y(\beta - \hat{\beta})^2 \right.\right.$$

$$\left.\left. + \; 2\, M_{xy}\, (\alpha - \hat{\alpha})(\beta - \hat{\beta})\right]\right\} . \tag{22}$$

By invoking the central limit theorem Eq. (22) is approximately valid even if the $p\,(\delta_i)$ are non-gaussian.

3.2 Remarks

The phase reference is immaterial, provided of course it is the same for all elements. In fact, by using the obvious identities (A4) of Appendix A it is found that $\hat{\alpha}$ and $\hat{\beta}$ remain unchanged if in (16-17) every ψ_i is replaced by:

$$\psi_i' = \psi_i + \psi_c ,$$

with ψ_c a constant phase (independent of the index i). For example the phases can be measured with respect to the output of an auxiliary sensor whose position with respect to the array elements is immaterial and need not even be known. Alternatively and more conveniently the phases can be measured with respect to a particular array element, say the one indexed as 1. This leads to an estimator whose expression, while formally different, is completely equivalent to (16-17) because its expression is obtained from (16-17) through analytic manipulation. In fact the following identity holds as a consequence of (A4):

$$\sum_{i=1}^{N} \psi_i \left[(x_i - \bar{x}) M_y - (y_i - \bar{y}) M_x\right]$$

$$= \sum_{i=2}^{N} \psi_i \left[(x_i - \bar{x}) M_y - (y_i - \bar{y}) M_x\right] - \sum_{i=2}^{N} \psi_1 \left[(x_i - \bar{x}) M_y - (y_i - \bar{y}) M_x\right] ,$$

and thus (16) can be written:

$$\hat{\alpha} = \frac{\sum_{i=2}^{N} (\psi_i - \psi_1) \left[(x_i - \bar{x}) M_y - (y_i - \bar{y}) M_x\right]}{M_x M_y - M_{xy}^2} , \qquad (23)$$

and a parallel expression can be established for $\hat{\beta}$. Since (23) is identical to (16) all the statistical properties established for (16) evidently hold for (23) also. Of course in this case, unlike the scheme using an additional element for phase reference, the position of the element index by 1 plays a fundamental role in the algorithm (23) because it appears in the quantities (1-5).

Since the phases are measured modulo 2π there is an inherent ambiguity in the phase measurement. This can be eliminated, however, by choosing the spacings of some elements small enough to yield an unambiguous phase difference between contiguous elements. The technique is in principle well known[4] and a detailed study of this point is outside the scope of this paper. However a brief discussion will be made in Appendix D in connection with the numerical example of Section 4.

It is reasonable to assume that in practical cases the array will have at least one axis of symmetry, x or y. This assumption makes

$$M_{xy} = 0$$

[see (5)]. The two estimates $\hat{\alpha}$ and $\hat{\beta}$ are statistically independent [as shown by (22) and thus (16) takes the form:

$$\hat{\alpha} = \frac{1}{k} \frac{\sum_{i=1}^{N} \psi_i (x_i - \bar{x})}{M_x} . \tag{24}$$

A similar form holds for $\hat{\beta}$ (with y replacing x). The rms estimation error is:

$$\sigma_{\alpha}^{2} = \frac{\sigma_{\delta}^{2}}{k^2} \frac{1}{M_x} . \tag{25}$$

with a parallel expression for σ_β^2. Expressions (24-25) are of course also those holding for linear arrays. The one-dimensional version of the modified algorithm (23) is

$$\hat{\alpha} = \frac{1}{k} \frac{\sum_{i=2}^{N} (\psi_i - \psi_1)(x_1 - \bar{x})}{M_x} . \tag{26}$$

It is clear from (19-20) or (25-26) that to increase accuracy for a given number of elements it is convenient to have the elements as sparse as possible, compatible with the need of avoiding phase ambiguities, in order to increase the "moments of inertia" M_x, M_y, directly affecting the accuracy of the estimates of the angular location.

4. LINEAR ARRAYS, COMPARISON WITH ACCURACY OF MULTIPLE INTERFEROMETERS

The processing scheme discussed in the preceding sections provides considerably more accuracy, that is, smaller rms angular estimation error, than does the "conventional" multiple interferometer technique. The comparison is made by considering the same linear array (that is with the same element locations and measurement errors) using two different processing schemes: the one proposed here and

that used in the "conventional" interferometric technique. In the latter the angular information is extracted through the algorithm described in Ref. 4, which with our notation takes the form:

$$\hat{\alpha}_m = \frac{1}{k} \frac{\sum_{i=2}^{N} (\psi_i - \psi_1)}{\sum_{i=2}^{N} (x_i - x_1)} . \tag{27}$$

The estimator (27) is unbiased. Its variance is found to be:

$$\sigma_m^2 = \frac{\sigma_\delta^2}{k^2} \frac{N(N - 1)}{\left| \sum_{i=2}^{N} (x_i - x_1) \right|^2} . \tag{28}$$

Expression (28) is always greater than (25). The result can be established simply by recalling that, for gaussian probability densities of the phase measurements, the rms error in angular location associated with the estimator (24) or (26) is the minimum possible for the given array geometry, because it meets the bound on rms estimation error in Appendix C. However, it can be found more directly by using simple arithmetic inequality.

In order to gain a better appreciation of the superiority of the estimator proposed in comparison with a multiple baseline interferometer scheme, consider the following particular case.

Example

Suppose that it is known a priori that the direction of the object is within a sector of ± 15° from the plane perpendicular to the array axis.* In order to avoid ambiguities we will chose the minimum spacing among the elements equal to 0.75 λ.

The distances between first and successive elements increase in geometric progression. In this way all the phase ambiguities are eliminated through a simple algorithm discussed in Appendix D. Thus if c is the distance of the first element from the origin of the x axis, the abscissae of the elements are:

*In practical cases this may mean that through an auxiliary antenna system the signal is blanked if incident from outside the sector of interest, the angular measuring system being activited only for targets within the sector.

$$x_1 = c\,,$$
$$x_2 = c + 0.75\,\lambda\,,$$
$$x_3 = c + 1.5\,\lambda\,,$$
$$x_4 = c + 3\,\lambda\,,$$
$$x_5 = c + 6\,\lambda\,,$$
$$x_6 = c + 12\,\lambda\,,$$
$$x_7 = c + 24\,\lambda\,.$$

The abscissa of the geometric center of mass is:

$$\bar{x} = \frac{\sum_{i=1}^{7} x_i}{7} = c + 6.75\,\lambda\,.$$

Its moment of inertia M_x with respect to the center of mass is, from (3):

$$M_x + 448.875\,\lambda^2\,.$$

Suppose that the rms error in the phase measurements at the element ports is:

$$\sigma_\delta = 0.5 \text{ radians} \simeq 28.648°\,.$$

From (25) the rms error in the object angular location is, in α units:

$$\sigma_\alpha^{\ 2} = \frac{\sigma_\delta^{\ 2}}{(2\pi)^2}\,\frac{1}{\sum_{i=1}^{N}\left(\frac{x_i - \bar{x}}{\lambda}\right)^2}\,.$$

Put

$$\alpha = \sin\theta,$$

with θ the angle from the array normal. The rms error of θ is maximum for $\theta = 15°$, the edge of the sector of interest, and is in milliradians.

$$\sigma_\theta = \frac{\sigma_\alpha}{\cos 15°}\,10^3 \simeq 3.89 \text{ mrad}\,. \tag{29}$$

Let us consider the rms error σ_θ' for the multiple interferometer scheme. By applying (28) we obtain $\sigma_\theta' = 11.3$ mrad. The rms error is substantially greater than for the optimum scheme.

If a different element is chosen as a reference the rms error will vary, still remaining, of course, greater than the value (29). For example, assume as a reference element the one located at $x = c + 24\lambda$. In such a case the rms estimation error becomes $\sigma_\theta'' = 10.6$ mrad, again greater than (29), as it must be. We recall that we have considered the ideal situation in which any angular interference is absent.

5. SPECULAR REFLECTION EFFECTS—A NUMERICAL EXPERIMENT

In the presence of coherent interference like specular reflection, the processing scheme here proposed, although not the theoretical optimum, on the basis of a limited numerical study seems to exhibit comparatively good performance with respect to a conventional multiple baseline interferometric system. A theoretical analysis seems impossible or at least exceedingly difficult, mainly because phase "errors" induced by specular reflection are, of course, highly correlated. A numerical experiment made for a particular case shows the superiority of the algorithm (24) or (26) proposed here with respect to the conventional multiple interferometer scheme (27).

Consider the array geometry discussed in the previous section. Suppose the array is vertically located on a reflecting terrain. The distance of the first and lowest element from the origin located on the ground is c. If θ is the elevation angle of the scattering object, and if R is the magnitude of the reflection coefficient, the output voltage at the ith element is:

$$V_i = e^{j\frac{2\pi}{\lambda}(x_i + c)\sin\theta}\left[1 + R e^{j\varphi_1 - j\frac{4\pi}{\lambda}(x_i + c)\sin\theta}\right], \tag{30}$$

where φ_1 is the phase of the reflection coefficient. Put

$$\varphi = \varphi_1 + \frac{4\pi}{\lambda}(c + \bar{x})\sin\theta\ . \tag{31}$$

The phases of the V_i are:

$$\psi_i = \frac{2\pi}{\lambda}(x_i + c)\sin\theta + tg^{-1}\left\{\frac{R\sin\left[\varphi - \frac{4\pi}{\lambda}(x_i - \bar{x})\sin\theta\right]}{1 + R\cos\left[\varphi - \frac{4\pi}{\lambda}(x_i - \bar{x})\sin\theta\right]}\right\}\ . \tag{32}$$

Inserting (32) into (24) or (26), one obtains for the error in the estimation of $\sin\theta$

$$\sin \hat{\theta} - \sin \theta = \frac{\sum_{i=1}^{N} (x_i - \bar{x})\, tg^{-1}\left\{\frac{R \sin [\varphi - \frac{4\pi}{\lambda}(x_i - \bar{x}) \sin \theta]}{1 + R \cos [\varphi - \frac{4\pi}{\lambda}(x_i - \bar{x}) \sin \theta]}\right\}}{\frac{2\pi}{\lambda} M_x} . \quad (33)$$

Assume the error is small. We obtain then:

$$\Delta\theta \simeq \frac{\sum_{i=1}^{N} (x_i - \bar{x})\, tg^{-1}\left\{\frac{R \sin [\varphi - \frac{4\pi}{\lambda}(x_i - \bar{x}) \sin \theta]}{1 + R \cos [\varphi - \frac{4\pi}{\lambda}(x_i - \bar{x}) \sin \theta]}\right\}}{\frac{2\pi}{\lambda} M_x \cos \theta} . \quad (34)$$

In (33), φ is a phase angle depending upon the distance of the array from the reflecting plane, the phase of the reflection coefficient and the element spacings. In practical cases φ is not known. Hence to assess the merit of the processing scheme, it is reasonable to compute the error as a function of θ for different values of φ and a large R. In Figures 1 and 3 the error has been plotted for R = 0.9 and values of φ equal to 0, π, and $\pm \pi/2$. Figures 2 and 4 show the error in a multiple interferometer system with the same array geometry and the same values of R and φ. It is apparent that for the scheme proposed here the maximum error is less and decreases more rapidly with increasing elevation than for conventional multiple interferometer systems. Although this may suggest that in general the algorithm (24) or (26) is superior to (27) even in the presence of specular reflection, more extensive computations are of course necessary to corroborate this contention.

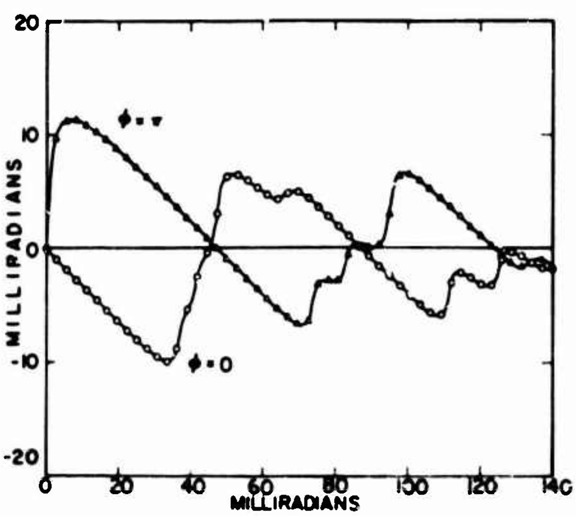

Figure 1. Proposed Scheme, Angle Estimate Error, R = 0.9, $\phi = \pi$, and $\phi = 0$

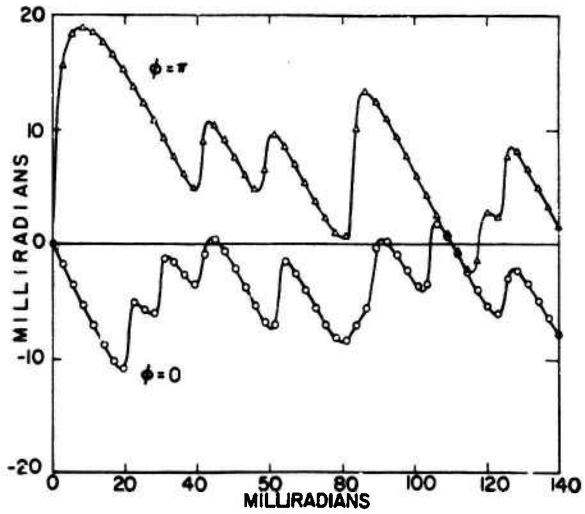

Figure 2. Multiple Interferometer, Angle Estimate Error, R = 0.9, $\phi = \pi$ and $\phi = 0$

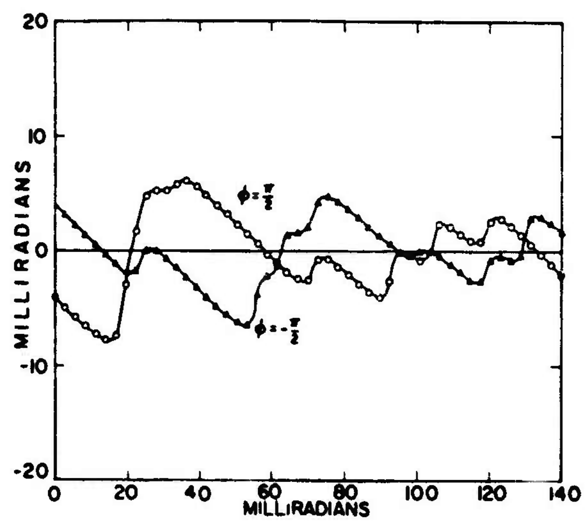

Figure 3. Proposed Scheme, Angle Estimate Error, R = 0.9, $\phi = \frac{\pi}{2}$, $\phi = -\frac{\pi}{2}$

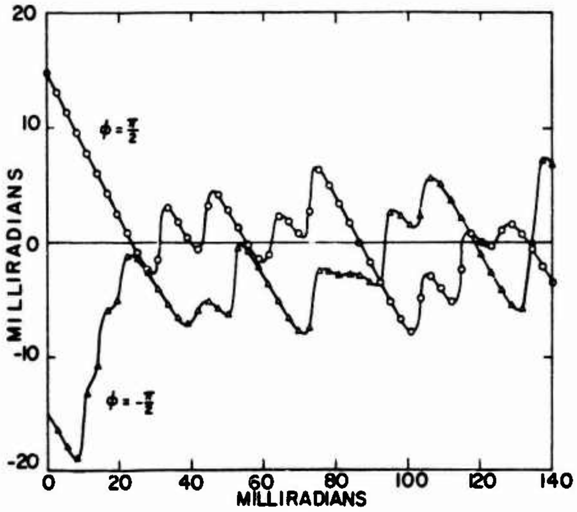

Figure 4. Multiple Interferometer, Angle Estimate Error, R = 0.9, $\phi = \frac{\pi}{2}$, $\phi = -\frac{\pi}{2}$

6. THERMAL NOISE

We consider again the situation of no angular interference. The presence of thermal noise can be accounted for by assuming that the output voltage of the ith sensor is a function of time described by:

$$v_i(t) = \left[A + m_i(t)\right] \cos(\omega_o t + \psi_i) + n_i(t) \sin(\omega_o t + \psi_i), \tag{35}$$

where ω_o is the center frequency and $m_i(t)$ and $n_i(t)$ are assumed to be narrowband gaussian processes. If B is the bandwidth (assumed rectangular), (35) can be sampled at a sampling rate of s/B. The phase and quadrature components of the samples are the real and imaginary parts of the complex quantities

$$V_i\left(\frac{s}{B}\right) = \sqrt{\left[A + m_i\left(\frac{s}{B}\right)\right]^2 + n_i^2\left(\frac{s}{B}\right)} \exp\left\{j\left[\psi_i + \gamma_i \frac{s}{B} + \omega_o \frac{s}{B}\right]\right\}, \tag{36}$$

where

$$\gamma_i\left(\frac{s}{B}\right) = tg^{-1} \frac{n_i\left(\frac{s}{B}\right)}{A + m_i\left(\frac{s}{B}\right)}. \tag{37}$$

Assume that, with probability close to unity:

$$A >> m_i(t), \qquad A >> n_i(t). \tag{38}$$

Then (36) yields

$$V_i\left(\frac{s}{B}\right) \simeq A \exp\left\{j\left[\psi_i + \frac{1}{A}\, n_i\left(\frac{s}{B}\right) + \omega_o\left(\frac{s}{B}\right)\right]\right\}, \tag{39}$$

which will be written concisely as:

$$V_{is} = A \exp\left[j\left(\varphi_{is} + \frac{\omega_o s}{B}\right)\right],$$

where the φ_i's are the phases of (39) and are independent random variables whose probability densities are induced by those of δ_i and $n_i(\frac{S}{B})$. The former is assumed gaussian and given by (11). The second is similarly given by:

$$P_n(n_{is}) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left(-n_{is}^2 / 2\sigma_n^{\,2}\right) \tag{40}$$

where we have concisely written:

$$n_{is} = n_i\left(\frac{S}{B}\right).$$

Let M = TB be the total number of samples, T being the observation time. The probability of the set of N x M phases φ_{is}, conditioned by the parameters α, β, μ, is:

$$p_t\left(\varphi_{11}, \varphi_{12} \cdots \varphi_{1M}, \varphi_{21} \cdots \varphi_{2M}, \varphi_{N1} \cdots \varphi_{NM} \,\middle|\, \alpha, \beta, \mu\right)$$
$$= \prod_{i=1}^{N} \prod_{s=1}^{TB} \int_{-\infty}^{-\infty} p_\delta\left(\psi_{is} - \mu - k\alpha\, x_i - k\,\beta\, y_i - \frac{n_{is}}{A}\right) p(n_{is})\, dn_{is}$$

or from (11) and (40), through some manipulation:

$$p_t\left(\varphi_{11}, \varphi_{12} \cdots \varphi_{1M}, \varphi_{21}, \varphi_{22} \cdots \varphi_{2M}, \varphi_{N1} \cdots \varphi_{NM} \,\middle|\, \alpha, \beta, \mu\right)$$
$$= \prod_{i=1}^{N} \prod_{s=1}^{TB} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\dfrac{\sigma_n^{\,2}}{A^2} + \sigma_\delta^{\,2}}} \exp\left\{-\frac{\left(\varphi_{is} - \mu - k\alpha x_i - k\beta y_i\right)^2}{2\left(\sigma_\delta^{\,2} + \dfrac{\sigma_n^2}{A^2}\right)}\right\}. \tag{41}$$

From (41) it is easily found by using the same procedure as in Section 4 that the MLE for $\hat{\alpha}$ and $\hat{\beta}$ are given by (16) and (17) with ψ_i replaced by:

$$\psi_i \leftarrow \frac{1}{M} \sum_{s=1}^{M} \varphi_{is}.$$

and the rms errors in the estimates are obtained from (19) and (20) by replacing σ_δ^2 by

$$\sigma_\delta^2 \leftarrow \sigma_\delta^2 + \frac{\sigma_n^2}{A^2\,TB} . \tag{42}$$

From (42) and (19-20) is found the very satisfactory result that if σ_δ^2 is negligible, the angular accuracy is inversely proportional to the time bandwidth product and the ratio between signal power and noise energy. This result can be put in an even more expressive form: If N_o is the noise power per cycle and E is the energy of the pulse of length T, the rms error in the angular estimation can be written in the following form:

$$\sigma_\alpha^2 = \left(\sigma_\delta^2 + \frac{1}{E/N_o}\right)\frac{k^{-2}}{M_x\,M_y - M^2_{xy}} ,$$

and thus the contribution to the rms error due to thermal noise (under the assumption of high signal noise ratio) is inversely proportional to the total energy in the pulse divided by the noise power per cycle.

7. CONCLUDING REMARKS

The ultimate accuracy provided by a linear or planar array of identical sensors in the determination of the direction of incidence of a plane wave has been studied for the ideal case of the absence of angular interference. If the phase measurements are affected by normally distributed errors, the optimum processor consists of a linear combination of the phases at the element output ports with weights depending upon the array geometry. It is shown that conventional multiple interferometer systems are substantially less accurate than the "optimum" processors introduced in this paper. The conclusion holds also in the presence of thermal noise.

Although the proposed scheme is not the theoretical optimum when terrain reflection is present, a very limited investigation has been conducted on the performance of a small array of sparse elements under that condition. For the particular case studied, the maximum errors are smaller than for multiple interferometer systems and, in most directions of the limited angular range investigated, the errors are smaller. How the scheme compares with other techniques in the presence of reflection and whether in this situation it is consistently superior to multiple interferometers is an open area of investigation of great practical interest.

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Appendix A

Form of Estimator for Normal Probability Density of Phase Errors

From (15) the estimate for μ is:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} \left(\psi_i - k\hat{\alpha}\, x_i - k\hat{\beta}\, y_i\right) \tag{A1}$$

which, introduced into (13-14) yields, by recalling (1) and (2):

$$\sum_{i=1}^{N} \psi_i\, x_i = \sum_{k=1}^{N} x_k \frac{1}{N} \sum_{i=1}^{N} \psi_i + k\hat{\alpha} \sum_{i=1}^{N} (x_i - \bar{x})\, x_i + k\hat{\beta} \sum_{i=1}^{N} (y_i - \bar{y})\, x_i\,, \tag{A2}$$

$$\sum_{i=1}^{N} \psi_i\, x_i = \sum_{k=1}^{N} y_k \frac{1}{N} \sum_{i=1}^{N} \psi_i + k\hat{\alpha} \sum_{i=1}^{N} (x_i - \bar{x})\, y_i + k\hat{\beta} \sum_{i=1}^{N} (y_i - \bar{y})\, y_i\,. \tag{A3}$$

Notice that it is evidently:

$$\sum_{i=1}^{N} (x_i - \bar{x}) = \sum_{i=1}^{N} (y_i - \bar{y}) = 0\,. \tag{A4}$$

On the basis of that we can recognize that (A2-A3) takes the form:

$$\sum_{i=1}^{N} \psi_i (x_i - \overline{x}) = k\hat{\alpha}\, M_x + k\hat{\beta}\, M_{xy},$$

$$\sum_{i=1}^{N} \psi_i (x_i - \overline{x}) = k\hat{\alpha}\, M_{xy} + k\hat{\beta}\, M_y,$$

from which the expressions (16-17) are immediately established.

Appendix B

Mean Value and Variance of the Estimates

Taking the average value of (16) and recalling (6-7) one gets:

$$E[\hat{\alpha}] = \frac{1}{k} \frac{\sum_{i=1}^{N} \left[\mu + k\alpha x_i + k\beta y_i\right] \left[(x_i - \bar{x}) M_x - (y_i - \bar{y}) M_y\right]}{M_x M_y - M_{xy}^2} \tag{B1}$$

or, through some manipulations, recalling (A4):

$$E[\hat{\alpha}] = \frac{1}{k} \frac{\alpha\left(M_y M_x - M_{xy}^2\right) + \beta\left[M_y M_{xy} - M_y M_{xy}\right]}{M_x M_y - M_{xy}^2} \tag{B2}$$

or

$$E[\hat{\alpha}] = \alpha . \tag{B3}$$

One finds the parallel result:

$$E[\hat{\beta}] = \beta . \tag{B4}$$

Equations (B3) and (B4) indicate that the estimates of the angular coordinates α, β are unbiased.

The variances of $\hat{\alpha}$ and $\hat{\beta}$ provide the rms error in the angular estimation. By taking into account (9) we have

$$E\left[(\hat{\alpha} - \alpha)^2\right]$$

$$= \sigma_\delta^2 \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2 M_y^2 + \sum_{i=1}^{N} (y_i - \bar{y})^2 M_{xy}^2 - 2 M_x M_y \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{k^2 \left(M_x M_y - M_{xy}^2\right)},$$

that is expression (19). Similarly expression (20) is obtained. The covariance of the estimations of α and β, entering in the expression of the joint density (22), is found to be:

$$E\left[(\alpha - \hat{\alpha})\left(\beta - \hat{\beta}\right)\right]$$

$$= \frac{\sigma_\delta^2}{k^2} \frac{(M_x M_y + M_{xy}^2) \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y}) - M_y M_{xy} \sum_{i=1}^{N} (x_i - \bar{x})^2 - M_x M_{xy} \sum_{i=1}^{N} (y_i - \bar{y})^2}{\left(M_x M_y - M_{xy}^2\right)^2},$$

that is expression (21).

Appendix C

A Cramer-Rao Bound for Unbiased Estimates of Direction of Incidence

In this appendix we address ourselves to the following question. What is the maximum possible accuracy of target angular location that can be obtained by processing the observables ψ_i? We will consider only unbiased estimators, that is, those not having systematic errors. We restrict ourselves only to the ideal case of no angular interference (terrain reflection, and so forth) and no thermal noise. No great difficulty would be encountered however in taking into account the latter along the lines of Section 6. No hypothesis on the form of the phase error probability densities (8) is made. We assume, however, the phase errors to be uncorrelated and the probability density function to have everywhere a second derivative. To simplify notation rename the unknown parameters as follows:

$$c_1 \doteq \alpha \; ; \qquad c_2 \doteq \beta \; ; \qquad c_3 \doteq \mu \; . \tag{C1}$$

With this notation we can introduce simply the so-called Fisher information matrix whose elements J_{sk} are obtained as follows. Take the opposites of the averages of the second derivatives (with respect to the unknown parameters) of the logarithm of the conditional probabilities of the observables:[6]

$$J_{sk} = - E \left| \frac{\partial^2 \log p_t(\psi_1, \psi_2 \ldots \psi_N \mid c_1, c_2, c_3)}{\partial c_s \, \partial c_k} \right| \; . \tag{C2}$$

The averages in (C2) are taken over the observable quantities and the c_i are the "true" values of the parameters. However, it will be apparent in the sequel that their knowledge is not necessary to evaluate the quantities in (C2). According to the Cramer-Rao bound the rms errors σ_i^2 in the unbiased estimate of the parameter c_i satisfy the inequality:[6]

$$\sigma_i^2 \geq \text{cofactor of } J_{ii} . \tag{C3}$$

Our task is thus evaluating the set of J_{ik} given by (C1).

To begin with notice that no matter what the explicit form of $p(\delta_i)$ in (8), we have, from (10):

$$J_{sk} = -\sum_{i=1}^{N} E\left[\frac{\partial \delta_i}{\partial c_s} \frac{\partial \delta_i}{\partial c_k} \frac{d^2 \log p(\delta_i)}{d^2 \delta_i} + \frac{d \log p(\delta_i)}{d \delta_i} \frac{\partial^2 \delta_i}{\partial c_s \partial c_k}\right] . \tag{C4}$$

Also, since, from (6):

$$\delta_i = \psi_i - k(\alpha x_i + \beta y_i) , \tag{C5}$$

the second derivatives of δ_i do not contain the observables ψ_i. Thus the second term in (C4) is:

$$-\sum_{i=1}^{N} \frac{\partial^2 \delta_i}{\partial c_s \partial c_k} \int \frac{d \log p(\delta_i)}{d \delta_i} p(\delta_i) \, d\delta_i$$

$$= -\sum_{i=1}^{N} \frac{\partial^2 \delta_i}{\partial c_s \partial c_k} \int \frac{dp}{d\delta_i} \, d\delta_i = 0 .$$

Thus (C4) is simply:

$$J_{sk} = E\left[-\frac{d^2 \log p(\delta_i)}{d \delta_i^2}\right] \sum_{i=1}^{N} \frac{\partial \delta_i}{\partial c_s} \frac{\partial \delta_i}{\partial c_k} . \tag{C6}$$

The Fisher information matrix is thus explicitly written

$$\left\{J_{sk}\right\} = \begin{bmatrix} -N & -k\sum_{i=1}^{N} x_i & -k\sum_{i=1}^{N} y_i \\ -k\sum_{i=1}^{N} x_i & -k^2\sum_{i=1}^{N} x_i^2 & -k^2\sum_{i=1}^{N} y_i x_i \\ -k\sum_{i=1}^{N} y_i & -k^2\sum_{i=1}^{N} x_i y_i & -k^2\sum_{i=1}^{N} y_i^2 \end{bmatrix} E\left[\frac{d^2 \log p(\delta)}{d^2\delta}\right] . \tag{C7}$$

By using (C3), through some simple manipulations we obtain

$$\sigma_\alpha^2 \geq \left\{ E\left[-\frac{d^2 \log p(\delta)}{d^2\delta}\right]\right\}^{-1} \frac{M_{xy}}{M_x M_y - M_{xy}^2} , \tag{C8}$$

$$\sigma_\beta^2 \geq \left\{ E\left[-\frac{d^2 \log p(\delta)}{d^2\delta}\right]\right\}^{-1} \frac{M_x}{M_x M_y - M_{xy}^2} , \tag{C9}$$

where it is understood that the probability densities are conditioned to the true values of the unknown parameter. Equations (C8) and (C9) establish the maximum theoretically possible accuracy for an angular estimator, when the phases of the sensor measured voltages are subject to instrumental errors δ_i, whose probability density is $p(\delta_i)$. We recall that this result is valid for a single target (no interference or multipath), assuming identical patterns for all elements (not necessarily isotropic).

For normal error densities:

$$\left\{ E\left[-\frac{d^2 \log p(\delta)}{d\delta^2}\right]\right\}^{-1} = \sigma_\delta^2 . \tag{C10}$$

By comparison of (19-20) with (C8-C9) we reach the conclusion that for normal densities the phase slope reconstruction algorithm discussed in Section 3 is, in fact, the best possible processing scheme in the sense of minimizing, under our assumptions, the rms error of the estimate of the angular position of the target.

Appendix D

Elimination of Phase Ambiguity

In this brief discussion on the elimination of phase ambiguities we will mainly refer to the linear array discussed in Section 4.

Let us denote by

$$\Delta \psi_i = \psi_i - \psi_1 , \qquad (i = 2, 3 \ldots N) \tag{D1}$$

the observed phase difference between the element labled i and the first element. Consider now the quantities:

$$q_i = \left| \Delta \psi_i - \frac{x_i - x_1}{x_{i-1} - x_1} \Delta \psi_{i-1} \right| . \tag{D2}$$

Because of the very definition of $\Delta \psi_i$, it is

$$E[q_i] = 0 , \tag{D3}$$

where the averages are taken with respect to the phase measurement errors. Also it is not difficult to establish that the variances of the quantities (D3) are:

$$E[q_i^2] = \sigma_\delta^2 \left[1 + \left(1 - \frac{x_i - x_1}{x_{i-1} - x_1} \right)^2 + \left(\frac{x_i - x_1}{x_{i-1} - x_i} \right)^2 \right] . \tag{D4}$$

For i = 2, (D1) gives:

$$\Delta \psi_2 = \psi_2 - \psi_1 , \qquad \text{(D5)}$$

and we assume that the elements 2 and 1 are close enough to eliminate the possibility of ambiguity in the phase measurements, although, of course, measurement errors will be present. The value (D5) will belong to the interval $-\pi, \pi$. In the array considered in Section 4:

$$\frac{x_i - x_1}{x_{i-1} - x_1} = 2 , \qquad \text{(D6)}$$

for all i's. Thus we have

$$q_i = \left| \Delta \psi_i - 2 \Delta \psi_{i-1} \right| . \quad (i = 3 \ldots N) \qquad \text{(D7)}$$

Once (D5) has been determined in an unambiguous way, no ambiguity is involved in the determination of ψ_3 to be used in (26), if, among its possible values differing by multiples of 2π, we choose that for which

$$\left| \Delta \psi_3 - 2 \Delta \psi_2 \right| < \pi . \qquad \text{(D8)}$$

Once ψ_3 is determined, we may determine ψ_4 by repeating the procedure, that is, by requiring that for every i:

$$\left| \Delta \psi_i - 2 \Delta \psi_{i-1} \right| < \pi . \qquad \text{(D9)}$$

This leads in turn to the specification of the maximum phase-measurement error permissible. To show that, consider that from (D4) and (D6) we get, for our geometry:

$$E[q_i^2] = 4 \sigma_\delta^2 . \qquad \text{(D10)}$$

Assume, that with probability close to one, the quantity q_i never becomes greater than three times its rms value:

$$\left| \Delta \psi_i - 2 \Delta \psi_{i-1} \right| \leq 6 \sigma_\delta , \qquad \text{(D11)}$$

which, in conjunction with (D9), leads to the condition for a maximum allowable phase measurement error

$$\sigma_\delta \leq \frac{\pi}{6} . \tag{D12}$$